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Wigner–Eckart theorem and infinitesimal operators of group representations

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Abstract. The new notion of a tensor operator which transforms under a representation of the compact Lie group is introduced. This tensor operator is a linear space $\mathscr{V}$ of operators. In order to obtain the traditional tensor operator it is sufficient to choose a basis in the space $\mathscr{V}$. The Wigner–Eckart theorem is valid for our tensor operators. The notion of irreducibility and the equivalence relation for tensor operators are formulated. Necessary and sufficient conditions of irreducibility and of equivalence are proved. Using the Wigner–Eckart theorem, we give the method of evaluation of infinitesimal operators of the representations of compact and non-compact Lie groups. The new invariants of irreducible representations of compact groups are found. Their quantity is equal to that of independent Casimir operators.

1. Introduction

Tensor operators and the Wigner–Eckart theorem are very significant mathematical tools of theoretical physics. Therefore, any results relating to tensor operators are of great importance for applications.

Usually, by the term 'tensor operator' we mean the set of operators $\boldsymbol{T} = \{T_n, n = 1, 2, \ldots\}$ which transforms under some representation $g \to D_g^\lambda$ of a symmetry group G. This definition implies correspondence between the operators T_n and the orthonormal basis elements $|\lambda n\rangle$ of the carrier space V of $g \to D_g^\lambda$. We show in this paper that the concept of a tensor operator has a wider definition. We put forward the new notion of a tensor operator which does not depend on the choice of a basis in V. We understand that a tensor operator is the linear space $\mathscr{V}$ of operators, which act in a fixed Hilbert space H. Every operator $T \in \mathscr{V}$ corresponds to a fixed element of V. The Wigner–Eckart theorem is valid for this tensor operator. Every orthonormal basis in V gives (according to the correspondence between V and $\mathscr{V}$) a set of operators T_n, $n = 1, 2, \ldots$, which is a traditional tensor operator. Reduced matrix elements for the 'linear space' tensor operator are those for every tensor operator $\{T_n, n = 1, 2, \ldots\}$.

We give the definitions of irreducibility and equivalence of tensor operators, and prove necessary and sufficient conditions for tensor operators to be equivalent or irreducible. These conditions may be useful for applications of the Wigner–Eckart theorem (invariant wave equations, construction of representations of Lie superalgebras and so on).

The Wigner–Eckart theorem is a good tool for investigation of infinitesimal operators of group representations. Application of group representations in physics (elementary particle theory, nuclear and atomic physics) demands infinitesimal operators in different orthonormal bases (which correspond to different subgroup chains relating to physical problems). Using the Wigner–Eckart theorem we give formulae for infinitesimal operators of group representations, which allow us to evaluate their matrix elements in different bases. They express these matrix elements by means of some Clebsch–Gordan coefficients. In this way we obtain r new invariants of group representations, where r is the rank of the group. It is interesting to investigate these invariants. They may be of some importance for physics.

Let us note that some of our results were known for partial Lie groups. However, we give a new treatment of the results.

2. Tensor operator as a linear space

The definition of a tensor operator includes two representations of the symmetry group G. One of these representations realises the symmetry of the tensor operator. This representation and tensor operator act in the same Hilbert space H. The second representation of G acts upon the index of the tensor operator. Let us consider these representations.

Let $g \to T_g$ be a unitary representation of the compact group G in the Hilbert space H. In general, this representation is reducible. Therefore, H decomposes into an orthogonal sum of invariant irreducible subspaces

$$H = \sum_j \oplus m_j H_j. \tag{1}$$

Here $m_j H_j = H_j \oplus H_j \oplus \ldots \oplus H_j$ (m_j times). The irreducible representation of G in H_j will be denoted by $g \to T^j_g$. In order to distinguish between subspaces H_j with the same j we use an additional index s: H^s_j, $s = 1, 2, \ldots, m_j$.

Let $g \to D^\Lambda_g$ be an irreducible unitary representation of G in the space V. An orthonormal basis of V will be denoted by $|\Lambda n\rangle$, $n = 1, 2, \ldots, \dim D^\Lambda$. Then

$$D^\Lambda_{n'n}(g) = \langle \Lambda n | D^\Lambda_g | \Lambda n' \rangle \tag{2}$$

are matrix elements of $g \to D^\Lambda_g$.

Usually the tensor operator $\boldsymbol{T}^\Lambda \equiv \{T^\Lambda_n, n = 1, 2, \ldots, \dim D^\Lambda\}$, acting in the space H and transforming under the representation $g \to D^\Lambda_g$ of G, is defined as a set of operators T^Λ_n, for which

$$T_g T^\Lambda_n T_g^{-1} = \sum_{n'} D^\Lambda_{n'n}(g) T^\Lambda_{n'}. \tag{3}$$

The representation $g \to T_g$ of G realises the symmetry of the tensor operator $\boldsymbol{T}^\Lambda$. The linear span H' of the subspaces H^s_j has to be included in the domains of the operators T^Λ_n. Relation (3) has to be fulfilled in H'.

The components T^Λ_n of $\boldsymbol{T}^\Lambda$ are associated with the basis elements $|\Lambda n\rangle$ of the carrier space V of $g \to D^\Lambda_g$. Formula (3) shows that under the action $T^\Lambda_n \to T_g T^\Lambda_n T_g^{-1}$, the tensor operator $\boldsymbol{T}^\Lambda$ transforms in exactly the same way as the basis elements $|\Lambda n\rangle$ do. Thus, the usual definition of a tensor operator is connected with a fixed basis of V.

Let us take another orthonormal basis of V:

$$|\Lambda m\rangle' = \sum_n u_{nm} |\Lambda n\rangle. \tag{4}$$

The numbers u_{nm} constitute a unitary matrix. We can obtain the operators

$$\tilde{T}^{\Lambda}_{m} = \sum_{n} u_{nm} T^{\Lambda}_{n}. \tag{5}$$

Therefore,

$$T_g \tilde{T}^{\Lambda}_{m} T_g^{-1} = \sum_{n} u_{nm} T_g T^{\Lambda}_{n} T_g^{-1}. \tag{6}$$

Matrix elements $D^{\Lambda}_{m'm}(g)$ for the basis $|\Lambda m\rangle'$ are connected with matrix elements (2) by

$$D^{\Lambda}_{m'm}(g) = \sum_{nn'} u_{nm} \overline{u_{n'm'}} D^{\Lambda}_{n'n}(g).$$

Therefore,

$$\sum_{m'} D^{\Lambda}_{m'm}(g) \tilde{T}^{\Lambda}_{m'} = \sum_{nn'} \sum_{n''m'} u_{nm} \overline{u_{n'm'}} D^{\Lambda}_{n'n}(g) u_{n''m'} T^{\Lambda}_{n''}$$

$$= \sum_{nn'} u_{nm} D^{\Lambda}_{n'n}(g) T^{\Lambda}_{n'}. \tag{7}$$

Due to (3) the right-hand sides of (6) and (7) are equal. Hence

$$T_g \tilde{T}^{\Lambda}_{m} T_g^{-1} = \sum_{m'} D^{\Lambda}_{m'm}(g) \tilde{T}^{\Lambda}_{m'}. \tag{8}$$

Thus, a set of operators (5) is a tensor operator $\tilde{\boldsymbol{T}}^{\Lambda}$ transforming under the representation $g \to D_g$ of G.

Let us show that the tensor operators $\boldsymbol{T}^{\Lambda}$ and $\tilde{\boldsymbol{T}}^{\Lambda}$ have the same reduced matrix elements. Let $|jsi\rangle$, $i = 1, 2, \ldots, \dim H_j$, be an orthonormal basis of H^s_j. Then due to the Wigner–Eckart theorem (Barut and Raczka 1977, Butler 1975) matrix elements of the operators T^{Λ}_{n} can be represented as

$$\langle jsi | T_n | j's'i' \rangle = \sum_{r} \langle js \| \boldsymbol{T}^{\Lambda} \| j's' \rangle^r \langle ji | \Lambda n, j'i' \rangle^r \tag{9}$$

where $\langle \ldots | \ldots, \ldots \rangle^r$ are Clebsch–Gordan coefficients of G. Here r separates multiple irreducible representations $g \to T^j_g$ in the tensor product of the representations $g \to D^{\Lambda}_g$ and $g \to T^{j'}_g$. We have

$$\langle jsi | \tilde{T}^{\Lambda}_{m} | j's'i' \rangle = \sum_{n} u_{nm} \langle jsi | T^{\Lambda}_{n} | j's'i' \rangle$$

$$= \sum_{r} \langle js \| T^{\Lambda} \| j's' \rangle^r \sum_{n} u_{nm} \langle ji | \Lambda n, j'i' \rangle^r$$

$$= \sum_{r} \langle js \| T^{\Lambda} \| j's' \rangle^r \langle ji | \Lambda m, j'i' \rangle^r$$

where the last Clebsch–Gordan coefficient corresponds to the basis $|\Lambda m\rangle'$. Therefore, $\boldsymbol{T}^{\Lambda}$ and $\tilde{\boldsymbol{T}}^{\Lambda}$ have the same reduced matrix elements.

From the point of view of the usual definition of a tensor operator, $\boldsymbol{T}^{\Lambda}$ and $\tilde{\boldsymbol{T}}^{\Lambda}$ are different tensor operators. However, it is more natural to consider them as the same tensor operator, corresponding to different bases of the space V. Moreover, continuing this analogy we can consider the correspondence between vectors

$$v = \sum_{n} a_n |\Lambda n\rangle \in V \tag{10}$$

and operators

$$T_v^\Lambda = \sum_n a_n T_n^\Lambda. \tag{11}$$

A set of operators T_v^Λ constitutes a linear space (we denote it by $\mathscr{V}$). The correspondence $v \to T_v^\Lambda$ realises the isomorphism ϕ between V and $\mathscr{V}$. The mapping $T_v^\Lambda \to T_g T_v^\Lambda T_g^{-1}$ is a linear transformation $\mathscr{D}_g^\Lambda$ of $\mathscr{V}$. The correspondence $g \to \mathscr{D}_g^\Lambda$ is a representation of G in $\mathscr{V}$. The isomorphism ϕ realises equivalence of the representations $g \to D_g^\Lambda$ and $g \to \mathscr{D}_g^\Lambda$, $\mathscr{D}_g^\Lambda = \phi D_g^\Lambda \phi^{-1}$.

Now we can give the following definition of a tensor operator. The linear space $\mathscr{V}$ of operators (acting in the Hilbert space H) for which the mapping $T^\Lambda \to T_g T^\Lambda T_g^{-1}$, $T^\Lambda \in \mathscr{V}$, is the representation $g \to \mathscr{D}_g^\Lambda$ of G in $\mathscr{V}$ is called a tensor operator transforming under the representation $g \to \mathscr{D}_g^\Lambda$ of G. It follows from formulae (9)–(11) that

$$\langle jsi|T_v^\Lambda|j's'i'\rangle = \sum_r \langle js\|\boldsymbol{T}^\Lambda\|j's'\rangle^r \langle ji|v, j'i'\rangle^r \tag{12}$$

where the reduced matrix elements $\langle js\|\boldsymbol{T}^\Lambda\|j's'\rangle^r$ coincide with those of relation (9). Formula (12) is the Wigner–Eckart theorem for a tensor operator of new definition ('linear space' tensor operator).

It is clear that a 'linear space' tensor operator does not depend on a choice of basis $|\Lambda n\rangle$ of the space. Choosing the basis $|\Lambda n\rangle$ in V and taking the operators $T_n^\Lambda \equiv \phi|\Lambda n\rangle \in \mathscr{V}$ we obtain the tensor operator in the usual sense.

We can further generalise the definition of a tensor operator. Let $\mathscr{V}$ be a linear space of operators acting in the Hilbert space H, and $g \to T_g$ be a unitary representation of the group G, acting in H. Then $\mathscr{V}$ is called a tensor operator if for every $T \in \mathscr{V}$ we have $T_g T T_g^{-1} \in \mathscr{V}$.

The condition $T_g T T_g^{-1} \in \mathscr{V}$, $T \in \mathscr{V}$, means that the mapping $T \to T_g T T_g^{-1}$, $T \in \mathscr{V}$, is a representation of G in $\mathscr{V}$. Our tensor operator transforms under this representation.

It is clear that the condition $T_g T T_g^{-1} \in \mathscr{V}$, $T \in \mathscr{V}$, is very simple. Here we do not need to know the matrix elements $D_{nn'}^\Lambda(g)$ of the representatation of G. Moreover, our last definition gives a simple method of construction of tensor operators. Indeed, if we have the representation $g \to T_g$ of G in the Hilbert space H and the operator A in H, then taking the linear span $\mathscr{V}$ of the operators $T_g A T_g^{-1}$, $g \in G$, we obtain a tensor operator: $T_g \mathscr{V} T_g^{-1} \in \mathscr{V}$.

3. Wigner–Eckart theorem for different bases

The Wigner–Eckart theorem is proved for fixed bases $|jsi\rangle$ of the subspaces H_j^s of H. Reduced matrix elements in (9) are defined by

$$\langle js\|\boldsymbol{T}^\Lambda\|j's'\rangle^r = d^{-1} \sum_{ii'n} \langle jsi|T_n^\Lambda|j's'i'\rangle \langle \Lambda n, j'i'|ji\rangle^r \tag{13}$$

where d is the dimensionality of $g \to T_g^j$. Reduced matrix elements do not depend on i, i', and n, but their definition by formula (13) depends on the choice of bases $|jsi\rangle$ of H_j^s. Let us show that in reality reduced matrix elements do not depend on this choice. In other words, if $|jst\rangle'$ are new bases of spaces H_j^s then

$$\langle jst|T_n^\Lambda|j's't'\rangle = \sum_r \langle js\|\boldsymbol{T}^\Lambda\|j's'\rangle^r \langle jt|\Lambda n, j't'\rangle^r \tag{14}$$

where the reduced matrix elements coincide with those of formula (9). Let

$$|jst\rangle' = \sum_i a^j_{it} |jsi\rangle. \tag{15}$$

The numbers a^j_{it} (at fixed j) constitute a unitary matrix. It follows from (9) and (15) that

$$\begin{aligned}\langle jst|T^\Lambda_n|j's't'\rangle &= \sum_{ii'} \overline{a^j_{it}} a^{j'}_{i't'} \langle jsi|T_n|j's'i'\rangle \\ &= \sum_{ii'} \overline{a^j_{it}} a^{j'}_{i't'} \sum_r \langle js\|\boldsymbol{T}^\Lambda\|j's'\rangle^r \langle ji|\Lambda n, j'i'\rangle^r \\ &= \sum_r \langle js\|\boldsymbol{T}^\Lambda\|j's'\rangle^r \sum_{ii'} \overline{a^j_{it}} a^{j'}_{i't'} \langle ji|\Lambda n, j'i'\rangle^r \\ &= \sum_r \langle js\|\boldsymbol{T}^\Lambda\|j's'\rangle^r \langle jt|\Lambda n, j't'\rangle^r.\end{aligned}$$

Therefore, the reduced matrix elements in (9) and (14) coincide. It is clear that this assertion is valid if we take the operators T^Λ_v, $v \in V$, instead of the operators T^Λ_n, $n = 1, 2, \ldots, \dim D^\Lambda$.

4. Irreducibility and equivalence of tensor operators

Usually irreducibility of the tensor operator $\boldsymbol{T}^\Lambda \equiv \{T^\Lambda_n,\ n = 1, 2, \ldots\}$ is defined as irreducibility of the representation $g \to D^\Lambda_g$ of G. However, it is possible to give another definition of irreducibility, which is useful for applications.

The tensor operator $\boldsymbol{T}^\Lambda = \{T^\Lambda_v, v \in V\}$ is called *irreducible*, if a set of operators T_g, $g \in G$, and T^Λ_v, $v \in V$, is irreducible, i.e. if every bounded operator A, commuting with all operators T_g and T^Λ_v, is a multiple of the unit operator.

The tensor operators $\boldsymbol{T}^\Lambda = \{T^\Lambda_v, v \in V\}$ and $\tilde{\boldsymbol{T}}^\Lambda = \{\tilde{T}^\Lambda_v,\ v \in V\}$ transforming under the same representation $g \to D^\Lambda_g$ of G are called *equivalent* (or unitary equivalent) if there exists an invertible bounded (respectively unitary) operator U, transforming H into $\tilde{H}$, for which

$$UT_gU^{-1} = \tilde{T}_g, \qquad g \in G, \qquad UT^\Lambda_vU^{-1} = \tilde{T}^\Lambda_v, \qquad v \in V. \tag{16}$$

Here $g \to T_g$ and $g \to \tilde{T}_g$ are representations of G which realise the symmetry of $\boldsymbol{T}^\Lambda$ and $\tilde{\boldsymbol{T}}^\Lambda$, correspondingly. Let us note that here we do not suppose that the representations $g \to T_g$ and $g \to \tilde{T}_g$ are unitary.

We shall give necessary and sufficient conditions of irreducibility and equivalence of tensor operators. For their formulation we need some facts on the space H. This space is represented by formula (1). Let us represent the subspace m_jH_j in the form $m_jH_j = V_j \otimes H_j$. In order to define the space V_j we consider the basis $|jsi\rangle$, $i = 1, 2, \ldots$, $\dim H_j$, $s = 1, 2, \ldots, m_j$, of m_jH_j. We express these basis elements as $|js\rangle \otimes |ji\rangle = |js\rangle|ji\rangle$, where $|js\rangle$ are formal vectors, which define the orthonormal basis of the space V_j. The correspondence

$$|jsi\rangle \leftrightarrow |js\rangle|ji\rangle \tag{17}$$

realises an isomorphism of m_jH_j and $V_j \otimes H_j$. In the following we identify these two spaces. It is clear that the operators T_g, $g \in G$, acting on $|js\rangle|ji\rangle$, leave the vector $|js\rangle$ invariant. The space H can be represented as

$$H = \sum_j (V_j \otimes H_j). \tag{18}$$

Using relation (12) we can write

$$T_v^\Lambda|j_2s_2i_2\rangle=\sum_{j_1s_1i_1}\langle j_1s_1i_1|T_v^\Lambda|j_2s_2i_2\rangle|j_1s_1i_1\rangle$$
$$=\sum_{j_1s_1i_1r}\langle j_1s_1\|\boldsymbol{T}^\Lambda\|j_2s_2\rangle^r\langle j_1i_1|v,j_2i_2\rangle^r|j_1s_1i_1\rangle. \tag{19}$$

This formula and identification (17) allow us to introduce new operators which act on the spaces V_j and H_j. Namely, for each j_1, j_2 and r we introduce the operator $A(j_2j_1r)$ which transforms V_{j_2} into V_{j_1}:

$$A(j_2j_1r)|j_2s_2\rangle=\sum_{s_1}\langle j_1s_1\|\boldsymbol{T}^\Lambda\|j_2s_2\rangle^r|j_1s_1\rangle. \tag{20}$$

Thus, reduced matrix elements of the tensor operator T^Λ are matrix elements of $A(j_2j_1r)$. For each j_1, j_2 and r we also introduce the operator $W_v(j_2j_1r)$ which transforms H_{j_2} into H_{j_1}:

$$W(j_2j_1r)|j_2i_2\rangle=\sum_{i_1}\langle j_1i_1|v,j_2i_2\rangle^r|j_1i_1\rangle. \tag{21}$$

It follows from (17) and (19)–(21) that

$$T_v^\Lambda|j_2s_2\rangle|j_2i_2\rangle=\sum_{j_1r}\{A(j_2j_1r)\otimes W_v(j_2j_1r)\}|j_2s_2\rangle|j_2i_2\rangle. \tag{22}$$

It is clear that there is a finite number of summands in (22). They are defined by formula (21).

Theorem. Let tensor operators $\boldsymbol{T}^\Lambda=\{T_v^\Lambda, v\in V\}$ and $\tilde{\boldsymbol{T}}^\Lambda=\{\tilde{T}_v^\Lambda, v\in V\}$ transform under the same representation $g\to D_g^\Lambda$ of G and act in the Hilbert spaces H and $\tilde{H}$, correspondingly. Let the representations $g\to T_g$ and $g\to\tilde{T}_g$ realise the symmetry of these tensor operators. Then $\boldsymbol{T}^\Lambda$ and $\tilde{\boldsymbol{T}}^\Lambda$ are equivalent if and only if (a) the representations $g\to T_g$ and $g\to\tilde{\boldsymbol{T}}_g$ are equivalent; (b) there exists a uniformly bounded (in j) set of invertible operators U_j, associated with the summands in (19) and transforming V_j onto $\tilde{V}_j$, such that for every r

$$U_{j_1}A(j_2j_1r)U_{j_2}^{-1}=A(j_2j_1r); \tag{23}$$

(c) the operator U, which realises equivalence of the representations $g\to T_g$ and $g\to\tilde{T}_g$, has the form

$$U=\sum_j\oplus(U_j\otimes E_j), \tag{24}$$

where E_j is the unit operator in $H_j\equiv\tilde{H}_j$. For unitary equivalence of the tensor operators $\boldsymbol{T}^\Lambda$ and $\tilde{\boldsymbol{T}}^\Lambda$ the representations $g\to T_g$ and $g\to\tilde{T}_g$ have to be unitary equivalent, and the operators U_j have to be unitary.

Let us note that the formulation of the theorem assumes that the space $\tilde{H}$ is represented as

$$\tilde{H}=\sum_j\oplus m_jH_j=\sum_j\oplus(\tilde{V}_j\otimes H_j). \tag{25}$$

Uniform boundedness (in j) of the operators U_j means that $\sup_j\|U_j\|<a<\infty$, where a is a constant.

We shall prove the theorem for the case of equivalent tensor operators. The proof for unitary equivalence is the same. Let the tensor operators $\boldsymbol{T}^{\Lambda}$ and $\tilde{\boldsymbol{T}}^{\Lambda}$ be equivalent. Then there is an invertible bounded operator U for which the relations (16) are valid. From the first relation of (16) it follows that the representations $g \to T_g$ and $g \to \tilde{T}_g$ are equivalent. Then from the Schur lemma it follows that $U = \Sigma_j \oplus U'_j$, where $\{U'_j\}$ is a set of invertible uniformly bounded (in j) operators, transforming the space $m_j H_j \subset H$ onto the space $m_j \tilde{H}_j \subset \tilde{H}$. Since the operators T_g and $\tilde{T}_g$ act upon basis elements of the spaces $m_j H_j = V_j \otimes H_j$ and $m_j \tilde{H}_j = \tilde{V}_j \otimes H_j$ by the same formulae, then due to the Schur lemma we have $U'_j = U_j \otimes E_j$, where U_j is an operator from V_j to $\tilde{V}_j$. Therefore, the relation (24) is proved. Since $\|U'_j\| = \|U_j\|$ then a set of the operators U_j is uniformly bounded in j. It is clear that the operators U_j are invertible. Substituting the expressions (22) for the operators T_v^{Λ} and $\tilde{T}_v^{\Lambda}$ and the expression (24) for the operator U into the second relation of (16) we obtain relation (23). Thus, necessity is proved. Sufficiency is proved in the same way by reversing the order of the reasoning. The theorem is proved.

Theorem. The tensor operator $\boldsymbol{T}^{\Lambda} = \{T_v^{\Lambda}, v \in V\}$ is irreducible if and only if for every uniformly bounded (in j) set of operators C_j acting in the spaces V_j, fulfilment of the relations

$$C_{j_1} A(j_2 j_1 r) = A(j_2 j_1 r) C_{j_2} \tag{26}$$

implies the equalities $C_j = aE_j$, where E_j is the unit operator in V_j and a is a constant.

Proof of this theorem is similar to that of the previous theorem. Let $\boldsymbol{T}^{\Lambda} = \{T_v^{\Lambda}, v \in V\}$ be an irreducible tensor operator. Let C_j be operators acting in different spaces V_j and satisfying the conditions of our theorem. Then we construct the operator $C = \Sigma_j \oplus (C_j \otimes E_j)$, acting in the space (18). Using the relations (22) and (26) it is easy to verify that C commutes with all operators T_v^{Λ}, $v \in V$, and T_g, $g \in G$. Since the tensor operator $\boldsymbol{T}^{\Lambda}$ is irreducible then $C = aE$, where a is a constant and E is the unit operator in H. Therefore, $C_j = aE_j$. Thus, necessity is proved. Sufficiency is proved in the same way by reversing the order of the reasoning. The theorem is thus proved.

5. Wigner–Eckart theorem and infinitesimal operators for representations of compact groups in different bases

Here the Wigner–Eckart theorem will be used in infinitesimal form. Instead of the representations $g \to D_g^{\Lambda}$ and $g \to T_g$ of the group G we consider the representations $a \to D_a^{\Lambda}$ and $a \to T_a$ of the Lie algebra $\mathscr{G}$ of G. Formula (3) in infinitesimal form can be written as

$$[T_a, T_n^{\Lambda}] = \sum_n D_{n'n}^{\Lambda}(a) T_{n'}^{\Lambda}. \tag{27}$$

Since $[\mathscr{G}, \mathscr{G}] \subset \mathscr{G}$, then the representation of $\mathscr{G}$ in the space $\mathscr{G}$ (adjoint representation) is defined. According to this representation the operator ad a,

$$(\mathrm{ad}\, a)x = [a, x] \in \mathscr{G}, \tag{28}$$

corresponds to the element a of $\mathscr{G}$. It is known that a scalar product can be introduced in $\mathscr{G}$ by means of the Cartan–Killing bilinear form. Let us choose an orthonormal basis $b_1, b_2, \ldots, b_n$ in $\mathscr{G}$. Let $D^{\mathrm{ad}}_{ii'}(a)$ be matrix elements of the adjoint representation of $\mathscr{G}$ with respect to this basis. These matrix elements coincide with the structure constants of $\mathscr{G}$. Relation (28) for basis elements $x = b_i$ can be written as

$$(\mathrm{ad}\, a)b_i \equiv [a, b_i] = \sum_{i'} D^{\mathrm{ad}}_{i'i}(a) b_{i'}. \tag{29}$$

Now we consider the finite-dimensional irreducible representation $a \to T^\lambda_a$ of $\mathscr{G}$. According to (29)

$$[T^\lambda_a, T^\lambda_i] = \sum_{i'} D^{\mathrm{ad}}_{i'i}(a) T^\lambda_{i'}, \tag{30}$$

where $T^\lambda_i = T^\lambda_{b_i}$. Relation (27) shows that the operators T^λ_i, $i = 1, 2, \ldots, n$, constitute the tensor operator $\boldsymbol{T}^{\mathrm{ad}}$, which transforms under the adjoint representation ad of $\mathscr{G}$. The representation $a \to T^\lambda_a$ of $\mathscr{G}$ realises the symmetry of this tensor operator. Thus, the space H of relation (1) in this case consists of one subspace H_j.

Let $|\lambda m\rangle$, $m = 1, 2, \ldots, \dim T^\lambda$, be a basis of the carrier space H of $a \to T^\lambda_a$. According to the Wigner–Eckart theorem

$$\langle \lambda m | T^\lambda_i | \lambda m' \rangle = \sum_q \langle \lambda \| \boldsymbol{T}^{\mathrm{ad}} \| \lambda \rangle^q \langle \lambda m | (\mathrm{ad}) i, \lambda m' \rangle^q. \tag{31}$$

As was shown in § 2, the reduced matrix elements $\langle \lambda \| \boldsymbol{T}^{\mathrm{ad}} \| \lambda \rangle^q$ are not dependent on the choice of orthonormal basis $|\lambda m\rangle$.

The formula (31) contains Clebsch–Gordan coefficients for the tensor product of the representations ad and $a \to T^\lambda_a$. Due to formula (11) of Klimyk (1968) this tensor product contains the representation $a \to T^\lambda_a$ with multiplicity $n_\lambda \leq r$, where r is the rank of $\mathscr{G}$. Moreover, there is an infinite number of representations $a \to T^\lambda_a$ for which $n_\lambda = r$. Therefore, we can formulate the following theorem.

Theorem. Every irreducible representation $a \to T^\lambda_a$ of $\mathscr{G}$ is characterised by r numbers $\langle \lambda \| \boldsymbol{T}^{\mathrm{ad}} \| \lambda \rangle^q$, $q = 1, 2, \ldots, r$. Some of them may be equal to zero. Infinitesimal operators T^λ_i of the representation $a \to T^\lambda_a$ in any orthonormal basis $|\lambda m\rangle$ are given by formula (31).

Thus, the numbers $\langle \lambda \| \boldsymbol{T}^{\mathrm{ad}} \| \lambda \rangle^q$ are invariants of the representation $a \to T^\lambda_a$. Their quantity is equal to that of independent Casimir operators. However, in some cases the invariants $\langle \lambda \| \boldsymbol{T}^{\mathrm{ad}} \| \lambda \rangle^q$ are more useful than eigenvalues of Casimir operators for the representation $a \to T^\lambda_a$. For example, the relations of Casimir operators with matrix elements of infinitesimal operators are very complicated. The invariants $\langle \lambda \| \boldsymbol{T}^{\mathrm{ad}} \| \lambda \rangle^q$ are directly connected with these matrix elements.

Now the meaning of the invariants $\langle \lambda \| \boldsymbol{T}^{\mathrm{ad}} \| \lambda \rangle^q$ in the representation theory is not clear. Properties of these invariants will be derived in a separate paper, where their explicit expressions will be given for the groups $\mathrm{U}(n)$ and $\mathrm{SO}(n)$.

6. Infinitesimal operators for representations of compact and non-compact Lie groups in different bases

Now we consider infinitesimal operators from another point of view. Let G be a semisimple real non-compact Lie group and K its maximal compact subgroup. Let G be a complexification of G, and G_k its real compact form. Then G_k is of the same dimensionality as G. Moreover, $G_k \supset K$. Enumeration of the corresponding groups G and G_k can be found, for example, in Klimyk (1982) and Klimyk and Gruber (1979). Let $\mathscr{G}$ and $\mathscr{G}_k$ be Lie algebras of G and G_k, respectively. Let $\mathscr{K}$ be a Lie algebra of K. Then we have (Helgason 1962)

$$\mathscr{G} = \mathscr{K} + \mathscr{P}, \qquad \mathscr{G}_k = \mathscr{K} + \mathrm{i}\mathscr{P}, \qquad \mathrm{i} = \sqrt{-1}, \tag{32}$$

where sums are direct and $\mathscr{P}$ is a linear subspace in $\mathscr{G}$. These decompositions have the property (Helgason 1962)

$$[\mathscr{K}, \mathscr{P}] \subset \mathscr{P}, \qquad [\mathscr{K}, \mathrm{i}\mathscr{P}] \subset \mathrm{i}\mathscr{P}. \tag{33}$$

Therefore, we have the representations of $\mathscr{K}$ in $\mathscr{P}$ and in $\mathrm{i}\mathscr{P}$. Let p_s, $s = 1, 2, \ldots, k$, and p'_s, $s = 1, 2, \ldots, k$, be bases in $\mathscr{P}$ and $\mathrm{i}\mathscr{P}$, respectively. We denote the representations of $\mathscr{K}$ in $\mathscr{P}$ and in $\mathrm{i}\mathscr{P}$ by AD. (These representations are equivalent.) According to (33) we have

$$[a, p_s] = \sum_{s'} D^{\mathrm{AD}}_{s's}(a) p_{s'}, \qquad a \in \mathscr{K}, \tag{34}$$

$$[a, p'_s] = \sum_{s'} D^{\mathrm{AD}}_{s's}(a) p'_{s'}, \qquad a \in \mathscr{K}. \tag{35}$$

Now we consider the representation $a \to T^{\lambda}_a$ of the algebra $\mathscr{G}_k$. Operators which correspond to $a \in \mathscr{K}$, p_s and p'_s will be denoted by A, P_s and P'_s, respectively. It follows from (34) and (35) that

$$[A, P_s] = \sum_{s'} D^{\mathrm{AD}}_{s's}(a) P_{s'}, \qquad a \in \mathscr{K}, \tag{36}$$

$$[A, P'_s] = \sum_{s'} D^{\mathrm{AD}}_{s's}(a) P'_{s'}, \qquad a \in \mathscr{K}. \tag{37}$$

Therefore, $\boldsymbol{P} = \{P_s, s = 1, 2, \ldots, k\}$ and $\boldsymbol{P}' = \{P'_s, s = 1, 2, \ldots, k\}$ are tensor operators which transform under the representation AD of $\mathscr{K}$. The reduction of the representation $a \to T^{\lambda}_a$ onto $\mathscr{K}$ realises the symmetry of these tensor operators. Let $H = \Sigma_j \oplus m_j H_j$ be a decomposition of H into invariant irreducible (with respect to $\mathscr{K}$) subspaces. Different subspaces H_j with the same j will be denoted by H^s_j, $s = 1, 2, \ldots, m_j$. Orthonormal basis elements of H^s_j will be denoted by $|jsi\rangle$. According to the Wigner–Eckart theorem we have

$$\langle jsi|P_s|j's'i'\rangle = \sum_q \langle js\|\boldsymbol{P}\|j's'\rangle^q \langle ji|(\mathrm{AD})s, j'i'\rangle^q, \tag{38}$$

$$\langle jsi|P'_s|j's'i'\rangle = \sum_q \langle js\|\boldsymbol{P}'\|j's'\rangle^q \langle ji|(\mathrm{AD})s, j'i'\rangle^q. \tag{39}$$

We consider that the representation operators for the subalgebra $\mathscr{K}$ are known. Therefore, we have to find them for $\mathscr{P}$ and $\mathrm{i}\mathscr{P}$ (cf formula (32)). In other words, we have to find the operators P_s and P'_s. They are defined by relations (38) and (39). To have them in an explicit form we have to find reduced matrix elements of (38) and (39). If infinitesimal operators P_s and P'_s are known for one choice of basis $|jsi\rangle$ then relations (38) and (39) define reduced matrix elements:

$$\langle js\|\boldsymbol{P}\|j's'\rangle^q = \sum_{si'} \langle jsi|P_s|j's'i'\rangle \langle (\mathrm{AD})s, j'i'|ji\rangle^q,$$

$$\langle js\|\boldsymbol{P}'\|j's'\rangle^q = \sum_{si'} \langle jsi|P'_s|j's'i'\rangle \langle (\mathrm{AD})s, j'i'|ji\rangle^q.$$

These reduced matrix elements can be used for obtaining the infinitesimal operators P_s and P'_s in any basis $|jsi\rangle'$. They can be obtained by means of Clebsch–Gordan coefficients for the basis $|jsi\rangle'$.

Thus, evaluation of an explicit form of infinitesimal operators P_s and P'_s in different bases is reduced to the evaluation of Clebsch–Gordan coefficients for different bases. There are many methods for the evaluation of Clebsch–Gordan coefficients. We mention here the recursive formula (47) of Klimyk (1982). The results of Klimyk (1979), Klimyk and Gruber (1979) and Gruber and Klimyk (1981) can be used for evaluation of partial Clebsch–Gordan coefficients.

7. Conclusion

We have generalised the notion of a tensor operator transforming under a representation of a compact group. This tensor operator is a linear space $\mathcal{V}$ of operators. The usual tensor operator can be obtained by any choice of a basis in $\mathcal{V}$.

We apply the Wigner–Eckart theorem to evaluate matrix elements of infinitesimal operators. This evaluation demands Clebsch–Gordan coefficients of partial type. Therefore, the problem is to evaluate these Clebsch–Gordan coefficients in different bases. Considering infinitesimal operators of representations of compact and non-compact groups, we see (cf formulae (38) and (39)) that they are of the same form.

We have obtained new invariants of irreducible representations of compact Lie groups. It is interesting to clarify their meaning in representation theory and for applications of group representations. Let us note that these invariants can be generalised to representations of non-compact semisimple Lie groups. The application of the Wigner–Eckart theorem for non-compact groups (Klimyk 1975) is needed.

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